N-63-4-3

EXAMINATION OF AN INVENTORY MODEL INCORPORATING PROBABILITIES OF OBSOLESCENCE

E. W. Barankin and J. Denny

Statistical Laboratory, University of California, Berkeley,
ONR 18 Technical Report to the Office of Naval Research
Prepared under Contract Nonr-222(43) (NR 042-036)

This is a working paper. Comments and suggestions are invited

EXAMINATION OF AN INVENTORY MODEL INCORPORATING PROBABILITIES OF OBSOLESCENCE

E.W. BARANKIN and J. DENNY

University of California, Berkeley

1. Introduction and description of the models. This paper is devoted to the numerical study of two models of the inventory problem. The first model is called the ordinary model. The second is called the obsolescence model, and describes an extension of the ordinary model in which the items in the inventory may become obsolescence at any stage. "Obsolescence" will mean that the item in question is no longer to be used and the stock on hand is to be disposed of. It will be seen that the ordinary model is a special case of the obsolescence model.

The intention of this piece of work is the modest one of providing an explicit comparison, in one particular case, of optimal inventory policies with and without the presence of obsolescence probabilities. Additional numerical studies will lend further insight into our obsolescence model, but above all, analytic studies are needed.

In Section 2 we set up the recursion relation for the ordinary model, and specify numerically the constants and component cost functions. In Section 3 we do the corresponding work for the obsolescence model, introducing there a specific probability distribution of time of obsolescence. The solutions of the problems of finding the optimal policies and optimum total cost functions in these two models are presented in Section 4.

On both models the same number $\underline{N}$ of time periods is fixed (N will be taken as 5 in our numerical work). These periods will be designated as $J_1, J_2, \ldots, J_N$, and the convention will be adopted that period N is the earliest in time, period N - 1 the succeeding period and period 1 the last period. Thus, the pertinent time diagram is as follows, if we label the inventory points from N to 0 with increasing time:

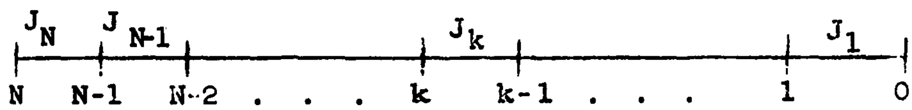

FIGURE I

The ordinary N-period model begins a period J_N with a primal stock of items in the inventory. Denote by x_N the size of the primal stock, which may be any real number in general but may be assumed to be nonnegative for this discussion. The primal stock can be increased by $y_N - x_N$ units where $y_N \geq x_N$. The quantity of item in the inventory after ordering, namely y_N, is called the starting stock for period N. During period N there will be a demand for ξ_N units and it will be assumed that the demand, which may be zero, always occurs after any replenishment $y_N - x_N$ to the primal stock x_N. For the following period, J_{N-1}, there will be a (possibly vanishing) left over stock from J_N, called the initial stock for period J_{N-1}, which will equal $y_N - \xi_N$, if this quantity is ≥ 0. But this quantity may be negative, and

if so it will represent a shortage in the preceding period. If indeed $y_N - \xi_N$ is negative, it will be assumed that any additional items obtained to replenish the initial stock in period J_{N-1} will first be consigned to the $-(y_N - \xi_N)$ unfilled units of demand from the preceding period. The starting stock y_{N-1} for period J_{N-1} will be the initial stock $y_N - \xi_N$ plus $y_{N-1} - (y_N - \xi_N)$, the amount by which the inventory is increased in period J_{N-1}. This procedure continues to period J_1 where the initial stock is $y_2 - \xi_2$, the remainder from period 2 and where the starting stock y_1 is initial stock $y_2 - \xi_2$ plus $y_1 - (y_2 - \xi_2)$, the replenishment to the inventory. If items remain in the inventory after the demand in period 1, i.e., if $y_1 - \xi_1 > 0$, the remainder will be sold for salvage.

There are various costs associated with the models. The cost of ordering quantities of the item to augment the primal and starting stocks is called the ordering cost. For both the regular model and the obsolescence model the ordering cost will consist of the cost of the items ordered plus a cost for placing the order, the latter being called the setup cost. The cost of failing to have an inventory at a fixed period large enough to meet the demand of that period is called the penalty cost. The cost of having a surplus at the end of a period after the demand of that period is called the holding cost. These costs also appear in the same fashion in the ordinary and obsolescence models. Salvage cost, which is a negative cost, is, in the ordinary model, the value of the remaining items if any at the end of period 1. This definition of salvage cost for the ordinary model will be modified for the obsolescence model. Finally, there is a discount factor.

On both the ordinary and obsolescence models the demands in the successive periods are assumed to be independent and to be identically distributed according to a known probability distribution. Demand is nonnegative.

The obsolescence model for N periods begins initially like the ordinary model. The primal stock x_N is increased to the starting stock y_N and subsequently there is a nonnegative demand ξ_N. After the demand ξ_N in period J_N but before the beginning of period J_{N-1}, obsolescence may occur according to some known probability. When this occurs, any remaining items are sold for salvage and no further orders or demands occur--the process stops. If obsolescence does not occur, then at the beginning of period J_{N-1} the initial stock $y_N - \xi_N$ is increased to the starting stock y_{N-1}. After the demand ξ_{N-1} in period J_{N-1} but before the beginning of period J_{N-2} obsolescence may occur with a certain probability. If obsolescence does occur here, then any remaining goods are sold for salvage. And so on, similarly.

It is clear from the above that salvage cost enters directly in each period in which the probability of obsolescence is not zero. When all probabilities of obsolescence are zero except for period 1, the obsolescence model becomes the ordinary model.

The component cost functions and the distribution of demand being known, the inventory problem is then to find an ordering policy for the N periods which will minimize the total expected discounted cost (but see Section 3). In the ordinary and obsolescence models the "optimal" policies are of the (s,S) type.

2. The recursion relation for the ordinary model. Let H_n denote the total discounted cost function for the n-period case, $n = 1, 2, \ldots$ The total cost sustained will depend on the primal stock, x_n, the successive demands in the n periods, $\xi_n, \xi_{n-1}, \ldots, \xi_1$, and the several starting stocks, $y_n, y_{n-1}, \ldots, y_1$. (As usual, the "initial stock," x_k, at t_k is the stock level resulting at the end of period $\bar{J}_{k+1}$, before stock-replenishment at t_k, and the "starting stock," aj_k, at t_k is the stock level at the beginning of $\bar{J}_k$, after stock-replenishment at t_k; thus, $y_k - x_k$ is the amount ordered for stock-replenishment at t_k.) Hence, the dependence of H_n is explicitly represented by $H_n(x_n; \xi_n, \xi_{n-1}, \ldots, \xi_1; y_n, y_{n-1}, \ldots, y_1)$.

If C denotes the replenishment cost function and l denotes the holding-shortage cost function--which two functions are the same for all periods--then evidently we have

$$(2.1) \quad H_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;y_n,y_{n-1},\ldots y_1) = C(y_n - x_n) + l(y_n - \xi_n) + \alpha H_{n-1}(y_n - \xi_n;\xi_{n-1},\xi_{n-2},\ldots,\xi_1;\ y_{n-1}, y_{n-2},\ldots,y_1)$$

for $n = 2, 3, \ldots$,

where α is the discount factor. For every period J_a, $k = 1,2,\ldots,n$, the functions C and l are given by

$$(2.2) \qquad C(z) = C_o \cdot z + \begin{Bmatrix} K, & \text{if } z > 0 \\ 0, & \text{if } z = 0 \end{Bmatrix},$$

$$(2.3) \qquad l(z) = \begin{cases} h \cdot z & \text{for } z \geq 0, \\ p \cdot (-z) & \text{for } z < 0, \end{cases}$$

where C_o, h and p are constant unit costs, and K is the setup cost for ordering.

The function H_1, the total cost function for the temporally last period, J_1, is determined with the assumption of disposal of left over items for a specified salvage value. If w denotes the salvage gain function, then we have

$$(2.4) \quad H_1(x_1;\xi_1;y_1) = C(y_1-x_1) + l(y_1-\xi_1) - w(y_1-\xi_1).$$

We take the function w to be characterized by a constant salvage value per unit of left over item, say w_o; thus, w is given by

$$(2.5) \qquad w(z) = \begin{cases} w_o \cdot z, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

Now (as usual) we consider the $y_1, y_2, \ldots, y_n$ in (2.1) replaced by functions $Y_1(x_1), Y_2(x_2), \ldots, Y_n(x_n)$ of the respective x_k, these functions to be determined according to an optimal principle, and thereby constituting the optimal policy. If we make this replacement, and for brevity set

$$\mathcal{Y}_n = (Y_n, Y_{n-1}, \ldots, Y_1),$$

(2.6)

$$H_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\mathcal{Y}_n) = H_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;Y_n(x_n), Y_{n-1}(x_{n-1}),\ldots,Y_1(x_1)),$$

then from (2.1) we get--on regarding the ξ_k as random variables--

$$(2.7)\quad \mathcal{E}H_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\mathcal{Y}_n) = \mathcal{E}[C(Y_n(x_n)-x_n)]$$

$$+\ \mathcal{E}\,[l(Y_n(x_n)-\xi_n)] + \qquad \xi n$$

$$+\ \alpha\mathcal{E}\left\{\mathcal{E}^{\xi n}[H_{n-1}(Y_n(x_n)-\xi_n;\xi_{n-1},\xi_{n-2},\ldots,\xi_1;\mathcal{Y}_{n-1}\right| ,$$

where $\mathcal{E}$ denotes expectation, and $\mathcal{E}^{\xi n}$ denotes conditional expectation given ξ_n.

The optimum principle is that (2.7) shall be minimized by suitable choice of $\mathcal{Y}_n$. If we denote this minimum by $C_n(x_n)$, then the customary argument gives, from (2.7),

$$(2.8)\quad C_n(x_n) = \min_{y\geq x_n}\left\{\mathcal{E}[C(y-x_n)] + \mathcal{E}[l(y-\xi_n)] + \alpha\mathcal{E}[C_{n-1}(y-\xi_n)]\right\}$$

<u>for n = 2, 3,....,</u>

and the minimizing value of y in (2.8) is the value of the optimal component function $Y_n(x_n)$.

The determination of $C_1(x_1)$ comes from (2.4); we find

$$(2.9)\quad C_1(x_1) = \min_{y\geq x_1}\left\{\mathcal{E}\,[C(y-x_1)] + \mathcal{E}\,[l(y-\xi_1)] - \mathcal{E}\,[w(y-\xi_1)]\right. \quad .$$

Together, (2.8) and (2.9) enable us to determine, successively for n = 1, 2,..., the optimal component functions $Y_k(x_k)$ and the optimal expected cost functions $C_k(x_a)$.

In our case at hand we are concerned with a five-period interval, and therefore we are interested in (2.9) and in (2.8) for n = 2, 3, 4 and 5.

In the present numerical study we specialize to the following values for the constants characterizing our total cost function:

$$
(2.10)\qquad \begin{cases} C_0 = \text{replenishment cost per unit of item} & = \frac{5}{6} \\ K = \text{set-up cost for ordering} & = 1 \\ h = \text{holding cost per unit of item} & = \frac{1}{2} \\ p = \text{penalty cost per unit of item} & = 6 \\ w_0 = \text{salvage value per unit of item} & = \frac{1}{3} \\ \alpha = \text{discount factor} & = 1 \end{cases}
$$

and we take the demands in the several periods to be independent and identically distributed, with density function φ given by

$$
(2.11)\qquad \varphi(\xi) = \begin{matrix} e^{-\xi}, & \xi \geq 0 \\ 0, & \xi < 0 . \end{matrix}
$$

We then have

$$
(2.12)\qquad \mathcal{E}[C(y - x)] = C(y - x) = \frac{5}{6}(y - x) + I_x(y)
$$

where

$$
(2.13)\qquad I_x(y) = \begin{cases} 1 & \text{if } y > x \\ 0 & \text{if } y \leq x, \end{cases}
$$

and

$$
(2.14)\qquad \mathcal{E}[L(y-\xi_k)] = \begin{cases} \frac{1}{2}\int_0^y (y-\xi)e^{-\xi}d\xi + 6\int_y^\infty (\xi-y)e^{-\xi}d\xi, & y > 0, \\ 6\int_0^\infty (\xi-y)e^{-\xi}d\xi, & y \leq 0 \end{cases}
$$

$$
= \begin{matrix} \frac{1}{2}(-1 + y + 13e^{-y}), & y > 0, \\ 6(1 - y), \; y \leq 0, \end{matrix} \qquad \text{(any } k\text{)}
$$

and

$$(2.15) \qquad \mathcal{E}\,[C_{n-1}(y-\xi_n)] = \int_0^\infty C_{n-1}(y-\xi)e^{-\xi}d\xi,$$

and finally,

$$(2.16) \qquad \mathcal{E}\,[w(y-\xi_1)] = \begin{cases} \frac{1}{3}\int_0^y (y-\xi)e^{-\xi}d\xi, & y > 0, \\ 0, \; y \leqq 0, \end{cases}$$

$$= \begin{cases} \frac{1}{3}(-1 + y + e^{-y}), & y > 0, \\ 0, \; y \leqq 0. \end{cases}$$

Inserting these evaluations into (2.9) we get

$$(2.17) \qquad C_1(x) = \min_{y \geqq x} \left[\frac{5}{6}(y-x) + I_x(y) + \begin{cases} \frac{1}{6}(-1 + y + 37e^{-y}), & y > 0 \\ 6(1-y), & y \leqq 0 \end{cases}\right]$$

and inserting them into (2.8) gives

$$(2.18) \qquad C_n(x) = \min_{y \geqq x} \left[\frac{5}{6}(y-x)+I_x(y)+\begin{cases} \frac{1}{2}(-1+y+13e^{-y}), & y > 0, \\ 6(1-y), & y \leqq 0 \end{cases} + \int_0^\infty C_{n-1}(y-\xi)e^{-\xi}d\xi\right].$$

With (2.17) we may now determine the optimum policy component Y_1 and the optimum expected cost function C_1. Then, iteratively, with (2.18) we determine Y_2, $C_2, \ldots,$ Y_5, C_5. By well-known arguments it follows that the optimum policy is an (s,S)-policy in each period. In Section 4 we give the results of our calculations, and we have there tabulated the optimal s_k and S_k, $k = 1, 2, \ldots, 5$.

3. The recursion relation for the obsolescence model. Let N denote the number of periods in which we are interested. This is specifically, in our present study, the number 5. For $n = 1,2, \ldots, N$, let π_n denote the probability that obsolescence occurs in the interval J_n. The latter eventualities are disjoint, by the nature of obsolescence. Furthermore since our inventory process comes to an end in any case after period J_1, we can consider the definition of obsolescence to be such that obsolescence certainly occurs in J_1 if it does not occur before. (Or, equivalently, we may be given the datum that obsolescence, priorly defined, certainly occurs within N periods, and thereby N is defined.) Thus, we have

$$\sum_{n=1}^{N} \pi_n = 1. \tag{3.1}$$

Let ω be a variable denoting the index of the period in which obsolescence occurs, For $n = 1, 2, \ldots, N$, let $\hat{H}_n$ denote the total discounted (to the inventory point n) cost function for the periods $J_n, J_{n-1}, \ldots J_1$. This function depends on the variables described in Section 2, but as well on the variable ω. And indeed the value of $\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;y_n,y_{n-1},\ldots,y_1;\omega)$ is determined by the values of x_n, ξ_n, $\xi_{n-1},\ldots,\xi_\omega$, y_n, $y_{n-1},\ldots,$ y_ω only.

The functions C, l and w, and the discount factor α are the same as in the ordinary model. Recalling that when obsolescence occurs in a particular period, any left-over quantity of the item is sold for salvage, we see that in the present case the relation between $\hat{H}_n$ and $\hat{H}_{n-1}$ is of the following form:

$$(3.2)\quad \hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;y_n,y_{n-1},\ldots,y_1;\omega) =$$

$$= \begin{cases} 0, & \underline{\text{if}}\ \omega > n, \\ c(y_n-x_n) + l(y_n-\xi_n) + \begin{cases} -w(y_n-\xi_n), & \underline{\text{if}}\ \omega = n, \\ \alpha\hat{H}_{n-1}(y_n-\xi_n;\xi_{n-1}, \end{cases} \\ \quad \xi_{n-2},\ldots,\xi_1;y_{n-1},y_{n-2},\ldots,y_1;\omega), & \underline{\text{if}}\ \omega \le n-1 \end{cases}.$$

<u>for n = 2, 3,..., N</u>

For $n = 1$, we have simply:

$$(3.3)\quad \hat{H}_1(x_1;\xi_1;y_1;\omega) = \begin{cases} 0, & \underline{\text{if}}\ \omega > 1, \\ C(y_1-x_1) + l(y_1-\xi_1) - w(y_1-\xi_1), & \underline{\text{if}}\ \omega = 1. \end{cases}$$

Let us denote the policy functions--to be determined by an optimality principle--by $\hat{Y}_1(x_1)$, $\hat{Y}_2(x_2),\ldots,\hat{Y}_N(x_N)$, and set

$$(3.4)\quad \begin{cases} \hat{\mathcal{Y}}_n = (\hat{Y}_n, \hat{Y}_{n-1},\ldots,\hat{Y}_1), \\ \hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega) = \hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1; \\ \qquad \hat{Y}_n(x_n),\hat{Y}_{n-1}(x_{n-1}),\ldots,\hat{Y}_1(x_1);\omega). \end{cases}$$

Now, for the present model the question presents itself whether the optimization principle should be to minimize, as is usual, the expectation

$$(3.5)\qquad \mathcal{E}\,\hat{H}_n(x_n;\xi_n, \xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega)$$

for $n = 1, 2,\ldots, N$, or alternatively, to minimize the conditional expectation

$$\mathcal{E}\left[\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega)\,\middle|\,\omega \leq n\right] \tag{3.6}$$

for each n. On general grounds the latter principle seems the more pertinent, the argument being that optimality considerations for the periods J_n, $J_{n-1},\ldots,J_1$ ought not to give any positive weighting to eventualities which, because they entail obsolescence before the period J_n, involve no behavior within the periods J_n, $J_{n-1},\ldots,J_1$. But in fact, in our specific model there is no difference between the two principles. This is so because the obsolescence probabilities π_k are fixed and the quantities which would get positive weighting under the first principle and not under the second are in fact all 0, so that the weighting is irrelevant. To see this more precisely, notice that by (3.2) and (3.3) we have, for all $n = 1, 2, \ldots, N$, that $\hat{H}_n = 0$ for $\omega > n$, and therefore (looking on the ξ_k and ω as random variables)

$$\begin{aligned}
\mathcal{E}\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega) &= \sum_{r=1}^{N}\pi_r\,\mathcal{E}\,\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;r) \\
&= \sum_{r=1}^{n}\pi_r\,\mathcal{E}\,\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;r) \\
&= \Big(\sum_{k=1}^{n}\pi_k\Big)\cdot\sum_{r=1}^{n}\frac{\pi_r}{\Big(\sum_{k=1}^{n}\pi_k\Big)}\,\mathcal{E}\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;r) \\
&= \Big(\sum_{k=1}^{n}\pi_k\Big)\,\mathcal{E}\left[H_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega\,|\,\omega \leq n\right]
\end{aligned} \tag{3.7}$$

<u>for n = 1, 2, ..., N.</u>

Thus, for each n, the expressions (3.5) and (3.6) differ only by a constant factor, and therefore the minimization of one is equivalent to the minimization of the other. (We are, of course, tacitly assuming in all our deliberations here that the π_k are suitably nonvanishing.)

Replacing the y_k by $\hat{Y}_k(x_k)$ in (3.2) and taking expectations, we get

$$(3.8)\quad \mathcal{E}\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega) = \left(\sum_{k=1}^{n}\pi_k\right)\left\{C(\hat{Y}_n(x_n)-x_n) + \mathcal{E}\left[l(\hat{Y}_n(x_n)-\xi_n)\right]\right\}$$

$$- \pi_n \mathcal{E}\left[\omega(\hat{Y}_n(x_n) - \xi_n)\right]$$

$$+ \alpha \sum_{r=1}^{n-1}\pi_r \mathcal{E}\hat{H}_{n-1}(\hat{Y}_n(x_n) - \xi_n;\xi_{n-1},\xi_{n-2},\ldots,\xi_1;\hat{\mathcal{Y}}_{n-1};r),$$

$$\underline{n = 2, 3, \ldots, N}.$$

This relation takes on a much more convenient form when expressed in terms of the conditional expectations (3.6). For brevity, let $\mathcal{E}^{(n)}$ denote the conditional expectation operator given $\omega \leq n$. Then, on dividing (3.8) through by $(\Sigma_{k=1}^{n}\pi_k)$ and utilizing (3.7)--both as it stands and with n replaced by n - 1--we find that (3.8) is equivalently expressed as:

$$(3.9)\quad \mathcal{E}^{(n)}\hat{H}_n(x_n;\xi_n,\xi_{n-1},\ldots,\xi_1;\hat{\mathcal{Y}}_n;\omega) = C(\hat{Y}_n(x_n) - x_n)$$

$$+ \mathcal{E}\left[l(\hat{Y}_n(x_n) - \xi_n)\right] - \mu_n \mathcal{E}\left[w(\hat{Y}_n(x_n) - \xi_n)\right]$$

$$+ \alpha(1 - \mu_n)\mathcal{E}^{(n-1)}\hat{H}_{n-1}(\hat{Y}_n(x_n)-\xi_n;\xi_{n-1},\xi_{n-2},\ldots,\xi_1;\hat{\mathcal{Y}}_{n-1};\omega),$$

$$\underline{\text{for } n = 2, 3, \ldots, N},$$

where

(3.10) $$\mu_n \overset{\text{def}}{=} \frac{\pi_n}{\sum_{k=1}^{n} \pi_k} .$$

Then, if $\hat{C}_n(x_n)$ denotes the minimum of (3.9), we have:

(3.11) $$\hat{C}_n(x) = \min_{y \geqq x} \left\{ C(y-x) + \mathcal{E}[l(y-\xi_n)] - \mu_n \mathcal{E}[w(y-\xi_n)] + \alpha(1-\mu_n)\mathcal{E}\hat{C}_{n-1}(y-\xi_n) \right\}$$

for $n = 2, 3, \ldots, N$.

For $n = 1$, we obtain

(3.12) $$\hat{C}_1(x) = \min_{y \geqq x} \left\{ C(y-x) + \mathcal{E}[l(y-\xi_1)] - \mathcal{E}[w(y-\xi_1)] \right\} .$$

For each n, the minimizing y for a given x is the optimal policy value $\hat{Y}_n(x)$.

Substituting into (3.11) and (3.12) the detailed functions and constants as specified in Section 2, we get:

(3.13) $$\hat{C}_1(x) = \min_{y \geq x} \left[\frac{5}{6}(y-x) + I_x(y) + \begin{cases} \frac{1}{6}(-1+y+37e^{-y}), & y > 0 \\ 6(1-y), & y \leqq 0 \end{cases} \right]$$

(notice that $\hat{C}_1$ is identical with C_1, given in (2.17)), and

(3.14) $$\hat{C}_n(x) = \min_{y \geq x} \left[\frac{5}{6}(y-x) + I_x(y) + \begin{cases} \frac{1}{2}(-1+y+13e^{-y}) - \frac{\mu_n}{3}(-1+y+e^{-y}), & y > 0 \\ 6(1-y), \ y \leqq 0 \end{cases} + (1-\mu_n)\int_0^\infty \hat{C}_{n-1}(y-\xi)e^{-\xi}d\xi \right] .$$

We see that the form of the problem here is the same as in the case of the ordinary model, there being simply the changes in coefficients in the recursion relation (3.14) due to the μ_n. Again the optimal policy is of the (s,S)-type for each period, and in Section 4 we present the optimal s_k and S_k.

We shall carry out our numerical study for the set of values of the π_k as tabulated below; we tabulate also the μ_k and the quantities

$$(3.15) \quad \lambda_k \overset{\text{def}}{=} \sum_{n=k}^{5} \pi_n = \text{probability that obsolescence occurs in one of the periods } J_5, J_4, \ldots, J_k.$$

TABLE I

k	π_k	μ_k	λ_k
1	$\frac{3}{8}$	1	1
2	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{5}{8}$
3	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{1}{2}$
4	$\frac{1}{8}$	$\frac{2}{11}$	$\frac{7}{16}$
5	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{16}$

graphically, these quantities look as follows:

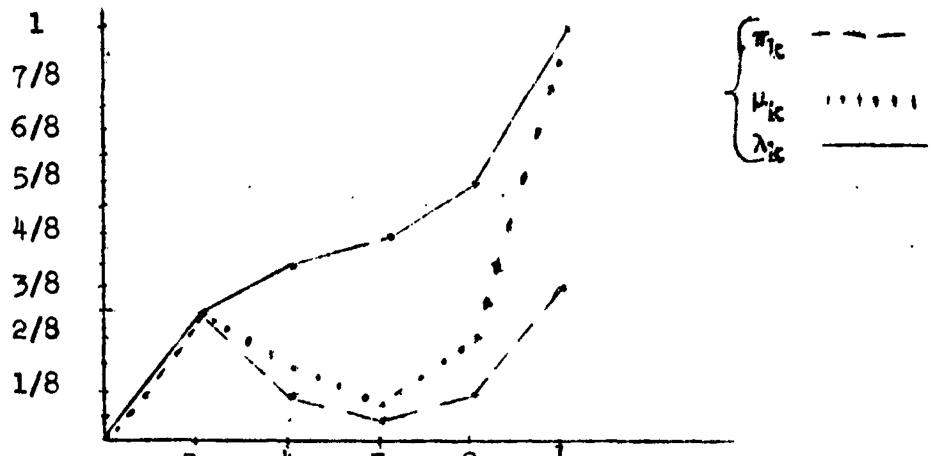

FIGURE II

4. The computational results. The results of carrying out the minimizations in (2.17), (2.18) and in (3.13), (3.14) are the following:

ORDINARY MODEL

$(s_1,S_1) = (0.67295, \ 1.81915)$

$$C_1(x) = \begin{cases} 3.65249 - \frac{5}{6}x & x < 0.67295 \\ -\frac{1}{6} + \frac{1}{6}x + 6\frac{1}{6}\,e^{-x} & x \geq 0.67295 \end{cases}$$

$(s_2,S_2) = (1.36731, \ 2.61030)$

$$C_2(x) = \begin{cases} 6.03523 - \frac{5}{6}x & x < 1.36731 \\ -\frac{5}{6} + \frac{2}{3}x + 10.47669\,e^{-x} \perp 6\frac{1}{6}xe^{-x} & x \geq 1.36731 \end{cases}$$

$(s_3, S_3) = (1.52891,\ 3.06648)$

$$C_3(x) = \begin{cases} 8.52142 - \frac{5}{6}x & x < 1.52891 \\ -2 + \frac{7}{6}x + 11.20586\, e^{-x} + 10.47669\, e^{-x}x \\ \quad + 3.08333\, e^{-x}x^2 & x \geqq 1.52891 \end{cases}$$

$(s_4, S_4) = (1.51388,\ 3.34470)$

$$C_4(x) = \begin{cases} 11.04205 - \frac{5}{6}x & x < 1.51388 \\ 8.85475 - \frac{1}{3}x + 6\frac{1}{2}\, e^{-x} & 1.51388 \leqq x < 1.52891 \\ -3\frac{2}{3} + 1\frac{2}{3}x + 17.10608\, e^{-x} + 11.20586\, e^{-x}x \\ \quad + 5.23834\, e^{-x}x^2 + 1.02776\, e^{-x}x^3 \\ & 1.52891 \leqq x \end{cases}$$

$(s_5, S_5) = (1.42970,\ 3.77837)$

$$C_5(x) = \begin{cases} 13.64619 - \frac{5}{6}x & x < 1.42970 \\ 11.37537 - \frac{1}{3}x + 6\frac{1}{2}\, e^{-x} & 1.42970 \leqq x < 1.5.388 \\ 8.68808 + \frac{1}{6}x + 5.4319\, e^{-x} \\ \quad + 6\frac{1}{2}\, e^{-x}x & 1.51388 \leqq x < 1.52891 \\ -5\frac{5}{6} + 2\frac{1}{6}x \quad 21.35778\, e^{-x} + 17.10608\, e^{-x}x \\ \quad + 5.60293\, e^{-x}x^2 + 1.74609\, e^{-x}x^3 \\ \quad + 0.25694\, e^{-x}x^4 & 1.52891 \leqq x \end{cases}$$

OBSOLESCENCE MODEL

$$(\hat{s}_1, \hat{S}_1) = (0.67295, \quad 1.81915)$$

$$\hat{C}_1(x) = \begin{cases} 3.65249 - \frac{5}{6}x & x < 0.67295 \\ -\frac{1}{6} + \frac{1}{5}x + 6\frac{1}{6}\, e^{-x} & x \geqq 0.67295 \end{cases}$$

$$(\hat{s}_2, \hat{S}_2) = (1.19718, \quad 2.4649)$$

$$\hat{C}_2(x) = \begin{cases} 5.49071 - \frac{5}{6}x & x < 1.19718 \\ -\frac{2}{3} + \frac{13}{24}x + 9.39913\, e^{-x} + 4.625\, e^{-x}x & x \geqq 1.19718 \end{cases}$$

$$(\hat{s}_3, \hat{S}_3) = (1.4024, \quad 2.8852)$$

$$\hat{C}_3(x) = \begin{cases} 7.49881 - \frac{5}{6}x & x \quad 1.4024 \\ -1.53702 + 0.94443x + 10.83690\, e^{-x} + 8.35474\, e^{-x}x + 2.05553\, e^{-x}x^2 & x \geqq 1.4024 \end{cases}$$

$$(\hat{s}_4, \hat{S}_4) = (1.26515, \quad 3.0228)$$

$$\hat{C}_4(x) = \begin{cases} 8.94219 - \frac{5}{6}x & x \quad 1.26515 \\ 6.37780 - 0.24270x + 6.43936\, e^{-x} & 1.26515 \leqq x < 1.4024 \\ -2.46966 + 1.21183x + 13.40940\, e^{-x} + 8.86653\, e^{-x}x + 3.41784\, e^{-x}x^2 + 0.56059\, e^{-x}x^3 & x \geqq 1.4024 \end{cases}$$

$$(\hat{s}_5, \hat{S}_5) = (1.11243, \ 2.82610)$$

$$\hat{C}_5(x) = \begin{cases} 9.15756 - \frac{5}{6}x & x < 1.11243 \\ 6.3284 + 0.65624x + 6.39585\, e^{-x} & 1.11243 \leqq x < 1.26515 \\ 4.15577 + 1.062304 + 6.66088 e^{-x} + 4.42706\, e^{-x}x & 1.26515 \leqq x < 1.4024 \\ -2.92635 + 2.06232x + 14.50318\, e^{-x} + 9.21896\, e^{-x}x + 3.04787 e^{-x}x^2 + 0.78525\, e^{-x}x^3 + 0.09635\, e^{-x}x^4 & x > 1.4024 \end{cases}$$

We summarize the critical numbers in the following table and graph:

TABLE II

k	Ordinary Model		Obsolescence Model	
1	0.67295	1.81915	0.67295	1.81915
2	1.36731	2.61030	1.19718	2.46490
3	1.52891	3.06648	1.40240	2.88520
4	1.51388	3.34470	1.26515	3.02280
5	1.42970	3.77837	1.11243	2.82610

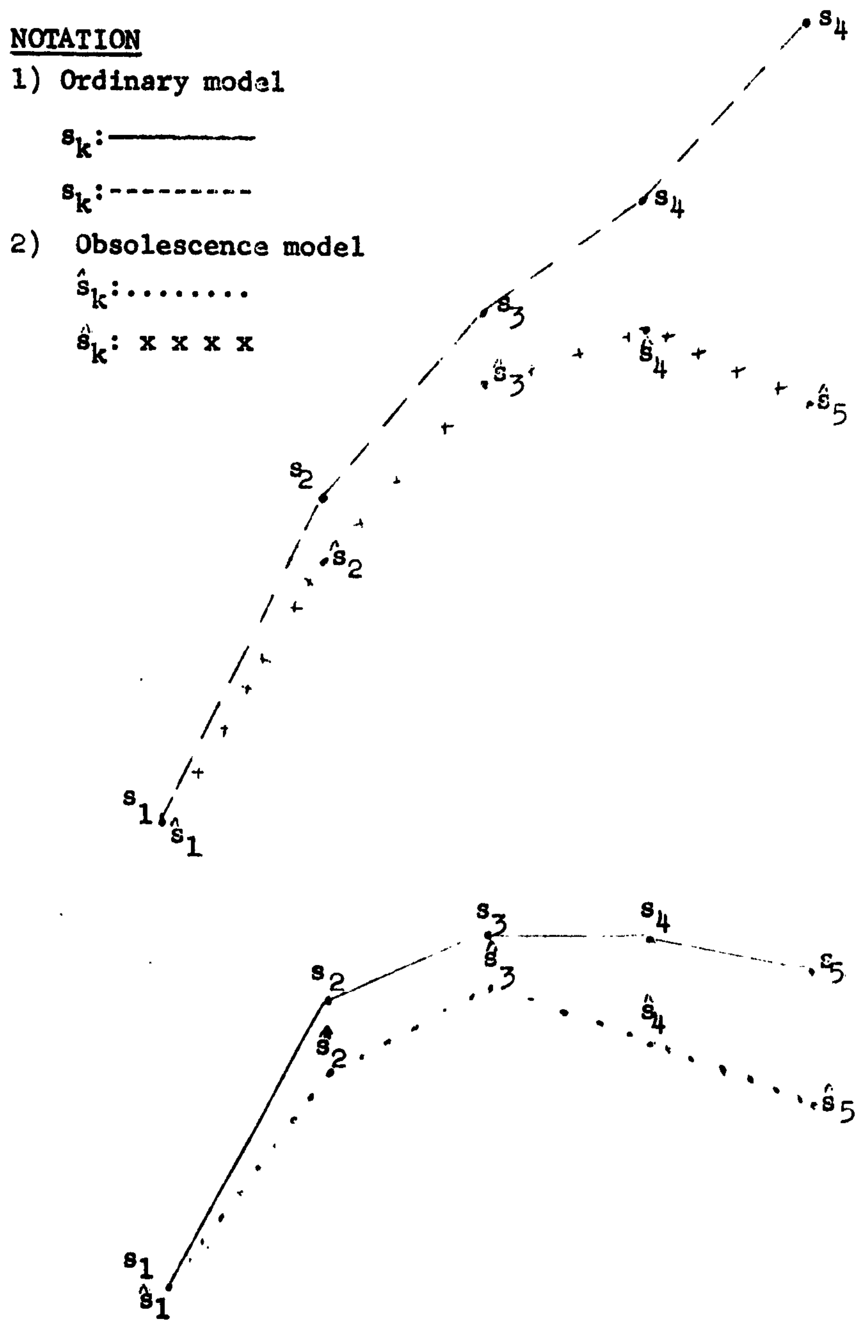
NOTATION
1) Ordinary model
s_k: ————
s_k: ---------
2) Obsolescence model
ŝ_k: ....
ŝ_k: x x x x
s1
ŝ1
s2
ŝ2
s3
ŝ3
s4
ŝ4
s4
ŝ5
s1
ŝ1
s2
ŝ2
s3
ŝ3
s4
ŝ4
s5
ŝ5

FIGURE III

The domination of $\hat{S}_k$ by S_k and $\hat{s}_k$ by s_k for k = 2, 3, 4, 5 reflect the possibility of termination of the obsolescence model before period 1 and hence the need for smaller inventories. The agreement of the costs functions $C_1(x)$ and $\hat{C}_1(x)$ was noted in (3.13) and is the reason for the agreement of $\hat{S}_1$ with S_1 and of $\hat{s}_1$ with s_1. The concave properties of the s_k-curve and the $\hat{s}_k$-curve are a consequence of the fact the ordering cost function C( ) is not convex (see (2.2)). The concavity of the $\hat{S}_k$-curve is reflected in part by the relatively high conditional probability of obsolescence in period 5 (see Table I).

UNIVERSITY OF CALIFORNIA - STATISTICAL LABORATORY

Distribution List for Unclassified Technical Reports

Contract Nonr-222(43)
Project (NR 042-036)

Head, Logistics and Mathematical Statistics Branch
Office of Naval Research
Washington 25, D.C. 3

Commanding Officer
Office of Naval Research Branch Office
Navy No. 100 Fleet Post Office
New York, New York 2

ASTIA Document Service Center
Arlington Hall Station
Arlington 12, Virginia 5

Office of Technical Services
Department of Commerce
Washington 25, D.C. 1

Technical Information Officer
Naval Research Laboratory
Washington 25, D.C. 6

Professor T. W. Anderson
Department of Mathematical Statistics
Columbia University
New York 27, New York 1

Professor Z. W. Birnbaum
Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle 5, Washington 1

Professor A. H. Bowker
Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California 1

Professor Ralph A. Bradley
Department of Statistics
Florida State University
Tallahassee, Florida 1

Dr. John W. Cell
Department of Mathematics
North Carolina State College
Raleigh, North Carolina 1

Professor Herman Chernoff
Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California 1

Professor W. G. Cochran
Department of Statistics
Harvard University
Cambridge, Massachusetts 1

Dr. C. Clark Cockerham
Institute of Statistics
North Carolina State College
Raleigh, North Carolina 1

Professor Cyrus Derman
Dept. of Industrial Engineering
Columbia University
New York 27, New York 1

Professor Benjamin Epstein
Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California 1

Professor W. T. Federer
Cornell University
Department of Plant Breeding
Biometrics Units
Ithaca, New York 1

Dr. R. J. Freund
Department of Statistics and Statistical Lab.
Virginia Polytechnic Institute
Blacksburg, Virginia 1

Professor H. P. Goode
Dept. of Industrial and
Engineering Administration
Cornell University
Ithaca, New York 1

Professor W. Hirsch
Institute of Mathematical Sciences
New York University
New York 3, New York 1

Professor Harold Hotelling
Associate Director
Institute of Statistics
University of North Carolina
Chapel Hill, North Carolina 1

Professor Oscar Kempthorne
Statistics Laboratory
Iowa State College
Ames, Iowa 1

Professor Gerald J. Lieberman
Applied Mathematics and
Statistics Lab.
Stanford University
Stanford, California 1

Dr. Arthur E. Mace
Battelle Memorial Institute
505 King Avenue
Columbus 1, Ohio 1

Professor J. Neyman
Department of Statistics
University of California
Berkeley 4, California 1

Dr. F. Oberhettinger
Department of Mathematics
Oregon State College
Corvallis, Oregon 1

Professor Herbert Robbins
Mathematical Statistics Dept.
Fayerweather Hall
Columbia University
New York 27, New York 1

Professor Murray Rosenblatt
Department of Mathematics
Brown University
Providence 12, Rhode Island 1

Professor H. Rubin
Department of Statistics
Michigan State University
East Lansing, Michigan 1

Professor I. R. Savage
School of Business
Administration
University of Minnesota
Minneapolis, Minnesota 1

Professor L. J. Savage
Statistical Research Laboratory
Chicago University
Chicago 37, Illinois 1

Professor W. L. Smith
Statistics Department
University of North Carolina
Chapel Hill, North Carolina 1

Professor Frank Spitzer
Department of Mathematics
University of Minnesota
Minneapolis, Minnesota 1

Dr. H. Teicher
Statistical Laboratory
Engineering Administration Bldg.
Purdue University
Lafayette, Indiana 1

Professor M. B. Wilk
Statistics Center
Rutgers-The State University
New Brunswick, New Jersey 1

Professor S. S. Wilks
Department of Mathematics
Princeton University
Princeton, New Jersey 1

Professor J. Wolfowitz
Department of Mathematics
Lincoln Hall
Cornell University
Ithaca, New York 1

BASIC DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL REPORTS

(LOGISTICS)

*Head, Logistics and Mathematical
Statistics Branch
Office of Naval Research
Washington 25, D. C. 3

*Commanding Officer
Office of Naval Research Branch
Office
Navy No. 100 Fleet Post Office
New York, New York 2

*ASTIA Document Service Center
Arlington Hall Station
Arlington 12, Virginia 10

*Office of Technical Services
Department of Commerce
Washington 25, D. C. 1

*Technical Information Officer
Naval Research Laboratory
Washington 25, D. C. 6

Commanding Officer
Office of Naval Research Branch
Office
346 Broadway
New York 13, New York
Attn: J. Laderman 1

Commanding Officer
Office of Naval Research Branch
Office
1030 East Green Street
Pasadena 1, California
Attn: Dr. A. R. Laufer 1

Professor Russell Ackoff
Operations Research Group
Case Institute of Technology
Cleveland 6, Ohio 1

Professor Kenneth J. Arrow
Serra House
Applied Mathematics and Statistics
Lab.
Stanford University
Stanford, California 1

Professor G. L. Bach
Carnegie Institute of Technology
Planning and Control of Industrial
Operations
Schenley Park
Pittsburgh 13, Pennsylvania 1

Professor A. Charnes
Mathematics Department
The Technological Institute
Northwestern University
Evanston, Illinois 1

Dr. David Gale
Department of Mathematics
Brown University
Providence 12, Rhode Island 1

Professor Warren M. Hirsch
New York University
25 Waverly Place
New York, New York 1

Dr. James R. Jackson
Management Sciences Research
Project
University of California
Los Angeles 24, California 1

Professor Samuel Karlin
Department of Mathematics
Stanford University
Stanford, California 1

Dr. C. E. Lemke
Mason Laboratory
Department of Mathematics
Rensselaer Polytechnic Institute
Troy, New York 1

Dr. W. H. Marlow
Logistics Research Project
The George Washington University
707 22nd Street, N. W.
Washington 7, D. C. 1

*Names with asterisk also appear on ONR list.

Professor Oskar Morgenstern
Economics Research Project
Princeton University
92 A Nassau Street
Princeton, New Jersey 1

Dr. R. Radner
Department of Economics
University of California
Berkeley, California 1

Professor Stanley Reiter
Department of Economics
Purdue University
Lafayette, Indiana 1

Professor A. W. Tucker
Department of Mathematics
Princeton University
Princeton, New Jersey 1

*Professor J. Wolfowitz
Department of Mathematics
Lincoln Hall
Cornell University
Ithaca, New York 1

Professor L. Hurwicz
School of Business Administration
University of Minnesota
Minneapolis 14, Minnesota 1

Mr. R. Simpson
Code W3
Bureau of Supplies and Accounts
Navy Department
Washington 25, D. C. 1

LCDR H. F. Mills, SC, USN
Ships Parts Control Center
Mechanicsburg, Pennsylvania 1

Mr. Robert Fickes
Aviation Supply Office
700 Robbins Avenue
Philadelphia 11, Pennsylvania 1

Mrs. Jane Olmer
Electronics Supply Office
Great Lakes, Illinois 1

Dr. George Suzuki
Navy Management Office
Navy Department
Washington 25, D. C. 1

LCDR Carl Dughi
Ordnance Supply Office
Mechanicsburg, Pennsylvania 1

Dr. Edward Berman
Operations Evaluation Group
Chief of Naval Operations
Navy Department
Washington 25, D. C. 1

Mrs. Marianne Smith
Office of the Director of
Logistics Plans
U. S. Air Force
Pentagon
Washington 25, D. C. 1

Mr. Bernard Rosenman
Frankford Arsenal
Philadelphia, Pennsylvania 1

Mr. Edward E. Oppenheim
Office, Assistant Secretary of
Defense
(Supply and Logistics)
Department of Defense
Washington 25, D. C. 1

Mr. Robert Brown
Apt. 106H
Eagle Heights
Madison 5, Wisconsin 1

Professor Robert Solow
Massachusetts Institute of
Technology
Cambridge, Massachusetts 1

Professor R. M. Thrall
University of Michigan
Ann Arbor, Michigan 1

Professor Max Woodbury
School of Engineering
New York University
New York, New York 1

Professor Herbert Galliher
Army Ordnance
Research Project
Massachusetts Institute of Technology
Cambridge, Massachusetts 1

Professor Herbert Simon
Carnegio Institute of Technology
Pittsburgh, Pennsylvania 1

Mr. Peter Winters
Carnegie Institute of Technology
Pittsburgh, Pennsylvania 1

Professor T. M. Whitin
School of Business Administration
University of California
Berkeley 4, California 1

Dr. Michel L. Balinski
Department of Mathematics
Princeton University
Princeton, New Jersey 1

Professor William J. Baumol
Princeton University
Princeton, New Jersey 1

Professor Martin J. Beckmann
Brown University
Providence, Rhode Island 1

Dr. C. B. Tompkins
University of California
Berkeley, California 1

Professor Herbert Solomon
Department of Statistics
Stanford University
Palo Alto, California 1

Professor William Feller
Fine Hall, Department of Mathematics
Princeton University
Princeton, New Jersey 1

Professor Merrill Flood
University of Michigan
231 W. Engineering Bldg.
Ann Arbor, Michigan 1

Professor Howard Raiffa
Harvard University
Boston 63, Massachusetts 1

Professor Ronald W. Shephard
University of California
Berkeley, California 1

Captain Charles Stein, SC, USN (Ret)
United Research, Inc.
808 Memorial Drive
Cambridge, Massachusetts 1

Dr. Francis Dresch
Stanford Research Institute
Menlo Park, California 1

Dr. Tibor Fabian
Lybrand, Ross Bros. and Montgomery
2 Broadway
New York 4, New York 1

Dr. Alan J. Hoffman
General Electric Company
570 Park Avenue
New York 22, New York 1

Mr. Robert G. Brown
Arthur D. Little, Inc.
35 Acorn Park
Cambridge 40, Massachusetts 1

Dr. Charles Zwick
RAND Corporation
Santa Monica, California 1

Dr. Ralph E. Gomory
IBM Research Center
Yorktown Heights, New York

Dr. Richard Bellman
RAND Corporation
1700 Main Street
Santa Monica, California 1

Director, Weapons System Evaluation Group
Attn: Mr. Bernard Sobin
Department of Defense
Pentagon
Washington 25, D. C.

Chief, Ordnance Supply Analysis Agency
Headquarters, Raritan Arsenal
Metuchen, New Jersey 1

Systems Research Group, Inc.
244 Mineola Blvd.
Mineola, L. I., New York 1

Corporation for Economic and Industry Research
General Analysis Division
11753 Wilshire Blvd.
Los Angeles 25, California 1

Technical Military Planning Operations
General Electric Company
Attn: Mr. Theodore Slattery
Santa Barbara, California 1

Planning Research Corporation
1333 Westwood Blvd.
Los Angeles 24, California 1

Stanford Research Institute
Menlo Park, California 1

MATHEMATICA Inc.
92 A Nassau Street
Princeton, New Jersey 1

United Research Incorporated
808 Memorial Drive
Cambridge, Massachusetts 1

Dunlap and Associates
429 Atlantic Street
Stamford, Connecticut 1

American Power Jet Company
Attn: Dr. George Chernowitz
705 Grand Avenue
Ridgefield, New Jersey 1

Cowles Foundation for Research in Economics
Yale University
Box 2125 Yale Station
New Haven, Connecticut 1

Operations Research Office
Attn: Dr. Ellis Johnson
Johns Hopkins University
6935 Arlington Road
Bethesda 14, Maryland 1

The Mitre Corporation
P. O. Box 31
Lexington, Massachusetts 1

George Washington University
Logistics Research Project
707 22nd Street, N. W.
Washington 7, D. C. 1

Chief, Bureau of Supplies and Accounts
Director, Advanced Logistics Research Division (Code W 3)
Navy Department
Washington 25, D. C. 20